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ON COMPLETELY TORSIONFREE MODULES

Dedicated to Professor Hirosi Nagao on his 60th birthday

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In this paper we shall consider some module theoretic generalizations of completely torsionfree rings ([4, p.91]), and study those rings whose modules are completely torsionfree (CTF). We refer for the definitions and basic properties concerning preradicals and torsion theories to [8, Chap. VI].

All rings occurring are associative and possess an identity element. All modules are unitary left modules. Let R be a ring. We write ${}_RM$ to indicate that M is an object in the category R-mod of all left R-modules. We denote by $E(A)$ the injective hull of ${}_RA$. A module M is said to be strongly prime (SP) if, for each left exact preradical σ for R-mod, either $\sigma(M) = 0$ or $\sigma(M) = M$. Also a module M is said to be 1SP if $\sigma(M) = 0$ for every proper left exact preradical σ for R-mod. SP-modules was studied in Beachy and Blair [1] and 1SP-modules was studied in [4, p.114].

Now, we shall define completely torsionfree modules as follows :

Definition. A module M is called *completely torsionfree* (CTF) if, for each left exact radical ρ for R-mod, either $\rho(M) = 0$ or $\rho(M) = M$. Further, M is called 1CTF (resp. 0CTF) if $\rho(M) = 0$ (resp. $\rho(M) = M$) for every proper (resp. nonzero) left exact radical ρ for R-mod.

Remark 1. (1) Let ρ be a preradical for R-mod. Then $\rho(R) = R$ if and only if $\rho = 1$, where 1 stands for the identity functor for R-mod. Hence a ring R is left CTF if and only if ${}_RR$ is 1CTF. Such a ring is also called left HRF in [2, p.160]. On the other hand, ${}_RR$ is 0CTF if and only if there exist only two left exact radicals for R-mod (i.e. R is left HRT ([2, p. 160])), or equivalently R is left ChC in the sense that, for all nonzero cyclic left R-modules C_1 and C_2, $\mathrm{Hom}_R(C_1, C_2) \neq 0$ holds ([4, p.96]). For characterizations of left CTF-rings, see [2, VI.1.E7] and [4, p.91], and for those of left ChC-rings, see [2, VI.2.1], [3], [4, p.96] and [7, Theorem 2.4].

(2) In [4, pp.117—119], we can find some connections between 1CTF-modules and CTF-rings.

(3) Hongan studied CTF-modules in [5, Theorem 1.3] and [6]. We notice that this result will also be obtained by using [2, I.6.4]. But we shall study in Theorem 5 the Hongan's result from our point of view. To do this and for further aims, we need next two lemmas.

Let ${}_RM$ be a fixed module. For each $x \in M$, we associate the left exact preradical σ_x and the left exact radical ρ_x for R-mod which are defined by

$$\sigma_x = \wedge\{\sigma \mid \sigma \text{ is a left exact preradical with } x \in \sigma(M)\}$$

and

$$\rho_x = \wedge\{\rho \mid \rho \text{ is a left exact radical with } x \in \rho(M)\}.$$

We also associate the set $\mathscr{L}_x$ of all left ideals I of R which contain $\mathrm{Ann}_R(Fx)$ for some finite subset F of R. Recall that there exists an order preserving correspondence between the left exact preradicals for R-mod and the left linear topologies on R, under which the left exact radicals correspond to the left Gabriel topologies.

Lemma 2 (cf. [4, pp. 89–90]). *Let x be an element of ${}_RM$. Then the set $\mathscr{L}_x$ is the smallest linear topology containing* $\mathrm{Ann}_R(x)$. *Moreover, the left exact preradical for R-*mod *corresponding to $\mathscr{L}_x$ is just σ_x.*

Proof. It is obvious that $\mathscr{L}_x$ is a filter. If $I \in \mathscr{L}_x$ and $s \in R$, then there exists a finite subset F of R with $\mathrm{Ann}_R(Fx) \subseteq I$, and so $\mathrm{Ann}_R(sFx) \subseteq (I: s)$. Hence $(I: s) \in \mathscr{L}_x$. Now, let $\mathscr{L}$ be an arbitrary left linear topology containing $\mathrm{Ann}_R(x)$. For each $I \in \mathscr{L}_x$, we have some finite subset F of R with $I \supseteq \mathrm{Ann}_R(Fx) = \bigcap_{r \in F}(\mathrm{Ann}_R(x) : r)$. By the assumption of $\mathscr{L}$, we have $I \in \mathscr{L}$.

Let $\mathscr{L}' \leftrightarrow \sigma_x$ and $\mathscr{L}_x \leftrightarrow \sigma'$ be the corresponding left linear topologies and left exact preradicals. Since $x \in \sigma_x(M)$, $\mathrm{Ann}_R(x) \in \mathscr{L}'$ and so $\mathscr{L}_x \subseteq \mathscr{L}'$. On the other hand, since $\mathrm{Ann}_R(x) \in \mathscr{L}_x$, we have $x \in \sigma'(M)$. Thus we obtain $\sigma_x \leqq \sigma'$ and hence $\mathscr{L}' \subseteq \mathscr{L}_x$. Therefore we must have $\mathscr{L}' = \mathscr{L}_x$ and $\sigma' = \sigma_x$.

Lemma 3. *Let x be an element of ${}_RM$. Then the smallest radical $\bar{\sigma}_x$ larger than σ_x is just ρ_x.*

Proof. Since $\bar{\sigma}_x$ is a left exact radical such that $x \in \sigma_x(M) \subseteq \bar{\sigma}_x(M)$, we have $\rho_x \leqq \bar{\sigma}_x$. Conversely, $x \in \rho_x(M)$ implies $\sigma_x \leqq \rho_x$ and so $\bar{\sigma}_x \leqq \rho_x$. Hence we have $\bar{\sigma}_x = \rho_x$.

Corollary 4. *Let x be an element of ${}_RM$, and let $\mathscr{G}_x$ denote the left Gabriel topology corresponding to ρ_x. Then a left ideal I of R belongs to $\mathscr{G}_x$ if and only if, for each proper left ideal J containing I, there exists $s \in R\backslash J$ such that $(J: s) \in \mathscr{L}_x$.*

Proof. Apply Lemma 3 with [8, Prop. VI. 5.4].

For a module ${}_RE$, we define the radical k_E by $k_E(X) = \bigcap\{\mathrm{Ker}(\alpha) \mid \alpha \in \mathrm{Hom}_R(X, E)\}$ for each $X \in R$-mod. It is well known that k_E is left exact whenever E is injective and every left exact radical has the form k_E for some injective module ${}_RE$. Now, we shall prove the next

Theorem 5. *The following properties of a nonzero module ${}_RM$ are equivalent:*

(1) *M is CTF.*

(2) *If K is a proper submodule of M and x is a nonzero element of M, then the next condition holds:*

(*) *There exist $y \in M\backslash K$ and a finite subset F of R such that* $\mathrm{Ann}_R(Fx) \subseteq (K: y)$.

(3) *If K is a proper submodule of M and x is a nonzero element of M, then* $\mathrm{Hom}_R(Rx, E(M/K)) \neq 0$.

(4) *If K is a proper submodule of M, then $k_{E(M/K)}(M) = 0$, or equivalently M is cogenerated by $E(M/K)$.*

(5) ([5]) *If K is a non-trivial submodule of M, then there exists $v \in K$ with* $\mathrm{Hom}_R(Rv, M/K) \neq 0$.

Proof. (1) $\Rightarrow$ (2). Note that the condition (*) is equivalent to the existence of $y \in M\backslash K$ with $(K: y) \in \mathscr{L}_x$, or equivalently $\sigma_x(M/K) \neq 0$ by Lemma 2. Assume that there exist a proper submodule K of M and a nonzero $x \in M$ such that $\sigma_x(M/K) = 0$. Since $\bar{\sigma}_x = \rho_x$ by Lemma 3, we have $\rho_x(M/K) = 0$. On the other hand, since $0 \neq x \in \rho_x(M)$, we must have $\rho_x(M) = M$ by (1), and so $\rho_x(M/K) = M/K$. This is a contradiction.

(2) $\Rightarrow$ (3). Let K be a proper submodule of M, and take a nonzero $x \in M$. By the assumption, we have a finite subset $F = \{s_1, \cdots, s_n\}$ of R with $\mathrm{Ann}_R(Fx) \subseteq (K: y)$. Put $u = (s_1x, \cdots, s_nx) \in (Rx)^{(n)}$. Consider a cyclic submodule $U = Ru$ of $(Rx)^{(n)}$. Then we get a nonzero homomorphism $f: U \to M/K$ given by $f(au) = ay + K$ ($a \in R$). Extend f to $\bar{f} \in \mathrm{Hom}_R((Rx)^{(n)}, E(M/K))$. Hence we see $\mathrm{Hom}_R(Rx, E(M/K)) \neq 0$.

(3) $\Leftrightarrow$ (4). Clear.

Mathematical Journal of Okayama University, Vol. 26 [1984], Iss. 1, Art. 21

(3) $\Rightarrow$ (5). Let K be a non-trivial submodule of M. Take a nonzero $x \in K$. By the assumption, $\mathrm{Hom}_R(Rx, E(M/K)) \neq 0$ and so there exists $v \in Rx$ with $\mathrm{Hom}_R(Rv, M/K) \neq 0$.

(5) $\Rightarrow$ (1). Assume that $\rho(M)$ is a non-trivial submodule of M for some left exact radical ρ. Then there exists $v \in \rho(M)$ with $\mathrm{Hom}_R(Rv, M/\rho(M)) \neq 0$. This is a contradiction, because Rv is ρ-torsion and $M/\rho(M)$ is ρ-torsionfree.

Lemma 6. *The following conditions are equivalent for an element x of $_RM$:*

(1) $\rho_x = 1$.

(2) *For each proper left ideal J of R, there exist $s \in R\backslash J$ and a finite subset $F = \{s_1, \cdots, s_n\}$ of R with* $\mathrm{Ann}_R(Fx) \subseteq (J : s)$. *(We may take $n = 1$).*

(3) *If $_RQ$ is a nonzero (cyclic) module, then* $\mathrm{Hom}_R(Rx, E(Q)) \neq 0$.

Proof. (1) $\Leftrightarrow$ (2). Since $\rho_x = 1$ if and only if $\mathscr{G}_x \ni \{0\}$, this is clear from Corollary 4.

(2) $\Rightarrow$ (3). Consider a nonzero cyclic module $Q = R/J$, where J is a proper left ideal of R. By the assumption, there exist $s \in R\backslash J$ and a finite subset F of R with $\mathrm{Ann}_R(Fx) \subseteq (J : s)$. Now, the same argument as in the proof (2) $\Rightarrow$ (3) of Theorem 5 enables us to have $\mathrm{Hom}_R(Rx, E(Q)) \neq 0$.

(3) $\Rightarrow$ (2). For each proper left ideal J of R, we have $\mathrm{Hom}_R(Rx, E(R/J)) \neq 0$ by using (3). Hence there exists a cyclic submodule Ru of Rx with a nonzero $f \in \mathrm{Hom}_R(Ru, R/J)$. Put $u = s_1x$ and $f(u) = s+J$ ($s_1, s \in R$). Since f is well defined, we have the condition (2).

Theorem 7. *The following properties of a nonzero module $_RM$ are equivalent:*

(1) *M is 1CTF.*

(2) *$\rho_x = 1$ for every nonzero $x \in M$.*

(3) *If $_RE$ is a nonzero injective module, then* $\mathrm{Hom}_R(Rx, E) \neq 0$ *for every nonzero $x \in M$.*

Proof. (1) $\Rightarrow$ (2). Let x be a nonzero element of M. Since ρ_x is a left exact radical with $x \in \rho_x(M)$, we have $\rho_x(M) \neq 0$ and so $\rho_x = 1$ by (1).

(2) $\Rightarrow$ (3). Let x be a nonzero element of M and E a nonzero injective module. By using Lemma 6, we see $\mathrm{Hom}_R(Rx, E) \neq 0$.

(3) $\Rightarrow$ (1). Let ρ be a proper left exact radical for R-mod. Then

there exists an injective module ${}_RE$ such that $\rho = k_E$. Since $\rho \neq 1$, we see $E \neq 0$. Now, for any $x \in \rho(M)$, we see from $k_E(Rx) = Rx$ that $\mathrm{Hom}_R(Rx, E) = 0$. Hence, $\rho(M) = 0$ by (3) and therefore M is 1CTF.

Theorem 8. *The following properties of a nonzero module ${}_RM$ are equivalent:*

(1) *M is 0CTF.*

(2) *If ${}_RE$ is an injective module with* $\mathrm{Hom}_R(M, E) \neq 0$, *then E is a cogenerator.*

(3) *If N is a proper submodule of M, then $E(M/N)$ is a cogenerator.*

(4) *If N is a proper submodule of M and ${}_RS$ is a simple module, then* $\mathrm{Hom}_R(S, M/N) \neq 0$.

Proof. (1) $\Rightarrow$ (2). Let E be an injective module with $\mathrm{Hom}_R(M, E) \neq 0$. Then $k_E(M) \neq M$ and so, by (1), $k_E = 0$ where 0 stands for the zero functor for R-mod. Hence, for every $X \in R$-mod, $k_E(X) = 0$ induces that X is cogenerated by E.

(2) $\Rightarrow$ (3). Let N be a proper submodule of M. By (2), $\mathrm{Hom}_R(M, E(M/N)) \neq 0$ implies that $E(M/N)$ is a cogenerator.

(3) $\Rightarrow$ (1). Let ρ be a left exact radical such that $\rho(M) \neq M$. Then, for an injective module ${}_RE$ such that $\rho = k_E$, we have $\mathrm{Hom}_R(M, E) \neq 0$. Take any $f(\neq 0) \in \mathrm{Hom}_R(M, E)$. By $M/\mathrm{Ker}(f) \cong \mathrm{Im}(f) \subseteq E$, we see $E(M/\mathrm{Ker}(f)) \subseteq$ E. Therefore, by (3), E must be a cogenerator and so we have $\rho = 0$.

(3) $\Leftrightarrow$ (4). Recall that an injective module ${}_RE$ is a cogenerator if and only if $\mathrm{Hom}_R(S, E) \neq 0$ for all simple module ${}_RS$.

Corollary 9. *A simple module ${}_RS$ is 0CTF if and only if $E(S)$ is a cogenerator.*

Corollary 10. *The following statements are equivalent for a ring R:*

(1) *Every simple left R-module is 0CTF.*

(2) *All simple left R-modules are isomorphic.*

Proposition 11. *The class of 0CTF-modules forms a hereditary torsion class. On the other hand, the class of 1CTF-modules forms a torsionfree class closed under injective hulls.*

Proof. By the definition, a module is 0CTF if and only if it belongs to

Mathematical Journal of Okayama University, Vol. 26 [1984], Iss. 1, Art. 21

the intersection of all nonzero hereditary torsion classes. Clearly this intersection forms a hereditary torsion class. The remaining part is proved in the same way.

Clearly every simple left R-module is CTF. We remark that every projective (more generally, torsionless) left R-module is CTF (or 1CTF) if and only if R is left CTF. In the next theorem we shall characterize those rings whose (injective) left modules are CTF. As proved in [4, p.118], a ring with a simple 1CTF-module is just of this type.

Theorem 12. *The following assertions are equivalent for a ring R:*

(1) *Every left R-module is 1CTF.*

(2) *Every injective left R-module is 1CTF.*

(3) *Every simple left R-module is 1CTF.*

(4) *Every left R-module is 0CTF.*

(5) *Every injective left R-module is 0CTF.*

(6) *Every projective left R-module is 0CTF.*

(7) *Every left R-module is CTF.*

(8) *Every injective left R-module is CTF.*

(9) ([7]) *Every nonzero injective left R-module is a cogenerator.*

(10) *There exist only two left exact radicals for R-mod.*

(11) ([4]) *R is left ChC.*

(12) ([3]) *R is left semiartinian and all simple left R-modules are isomorphic.*

Proof. (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (9). Let ${}_RE$ be a nonzero injective module. For each simple module ${}_RS$, since k_E is a proper left exact radical, we have $k_E(S) = 0$. Thus $\mathrm{Hom}_R(S, E) \neq 0$ and hence E is a cogenerator.

(4) $\Leftrightarrow$ (5) $\Rightarrow$ (6). Clear.

(6) $\Rightarrow$ (10). Since ${}_RR$ is 0CTF, this follows from Remark 1.

(7) $\Leftrightarrow$ (8). Clear.

(7) $\Rightarrow$ (12). Assume there exists a nonzero left R-module M with zero socle. We put the left exact preradical $\sigma = \mathrm{soc}$ and consider the left exact radical $\bar{\sigma}$. For a simple left R-module S, $\bar{\sigma}(M \oplus S) = S$ holds and so $M \oplus S$ is not CTF. Next, assume that S and T are non-isomorphic simple left R-modules. We put $\rho = k_{E(S)}$. Since $\rho(S) = 0$ and $\rho(T) = T$ by $\mathrm{Hom}_R(T, E(S)) = 0$, we have $\rho(S \oplus T) = T$, and hence $S \oplus T$ is not CTF.

(9) $\Leftrightarrow$ (10). This was proved in [7, Theorem 2.4].

(10) ⇔ (11). This was proved in [4, p.96].
(10) ⇔ (12). This was proved in [3, Proposition 2].
(10) ⇒ (1),(4) and (7). Clear.

Finally we shall give an example which distinguishes 0CTF-modules, 1CTF-modules and CTF-modules.

Example 13. (1) Every 0CTF-module is CTF, but the converse is not true. In fact, let R be a left CTF but not ChC ring (for example, a two-sided simple ring with zero socle). Then ${}_RR$ is 1CTF but not 0CTF by Remark 1.

(2) Every 1CTF-module is CTF, but the converse is not true. To see this, let R be a local but not right perfect ring (for example, the ring of formal power series over a field). Then the (unique) simple module S is 0CTF by Corollary 10, but since R is not left ChC ([4, p.96]), S is not 1CTF by Theorem 12.

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